

On the existence and uniqueness of solutions for fuzzy fractional differential equations

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Abstract

In this paper, we introduce and define a new metric on the space of fuzzy continuous functions in the fractional calculus. Regarding this metric and using the well-known Banach fixed point theorem, we provide some conditions that guarantee the existence and uniqueness of solution to a nonlinear fuzzy fractional differential equation in the proposed metric. Finally, two examples are given to illustrate the results.

2010 Mathematics Subject Classification. **34A08**. 34A12

Keywords. Fuzzy Liouville-Caputo's derivative, Metric space, Fuzzy fractional differential equation, Mittag-Leffler functions.

1 Introduction

Theory of differential equations is an important part of pure and applied mathematics which plays a prominent role in many disciplines including engineering, physics, economics and biology. One of the significant branches of theory of differential equations is fractional differential equations that in recent years, has received considerable attention not only in mathematical research but also in other applied sciences. In fact, Fractional differential equations are the development of differential equations to real order. These type of differential equations can be applied to many real-world field such as polymer physics, viscoelastic materials, viscous damping and seismic analysis see [3, 9, 11, 12, 13, 14, 18].

On the other hand, when one intends to analyze a real world phenomenon, it is also necessary to deal with uncertain factors. In this situation, the theory of fuzzy sets may be one of the best non-statistical or non-probabilistic approach, which leads us to investigate theory fuzzy fractional differential equations.

Recently, the topic of existence and uniqueness for the solutions to linear and nonlinear fuzzy fractional differential equations has been further investigated and discussed by many researchers in various aspects. For example, in [2] the existence and uniqueness of solutions of Riemann- Liouville fuzzy fractional differential equations has been proved by Arshad and in [16] the existence and uniqueness of solutions as well the approximate solutions to fuzzy fractional differential equations under the Liouville-Caputo H-differentiability has been studied by Salahshour and et al.. Furthermore, the existence and uniqueness of solutions for fuzzy fractional differential equations under the Liouville-Caputo generalized Hukuhara differentiability has been investigated by Allahviranloo et

al. in [1].

In this paper, we intend to propose a new metric on the space of fuzzy continuous functions by using Mittag-Leffler functions and we study and investigate the existence and uniqueness of solutions to nonlinear fuzzy fractional differential equations under fractional generalized H-differentiability in the sense of the Liouville-Caputo differentiability.

This paper organized as follows: A brief review on the well-known fuzzy concepts as well as fractional calculus are given in Section 2. Afterward in Section 3, a new metric for the space of fuzzy continuous functions is introduced and the main result of the paper i.e., the existence and uniqueness of solutions for nonlinear fuzzy fractional differential equations will be proved. Finally, two examples are given to illustrate the application of the result.

2 A brief review on fuzzy concepts and fractional calculus

2.1 Fuzzy Theory

Definition 2.1. [5] Let X be a nonempty set. A fuzzy set u in X is characterized by its membership function $u : X \rightarrow [0, 1]$. Then, for each $x \in X$ we interpret $u(x)$ as the degree of membership of the element x in the fuzzy set u : $u(x) = 0$ corresponded to non membership; $0 < u(x) < 1$ to partial membership; and $u(x) = 1$ to full membership.

Definition 2.2. [5] We denote by $\mathbb{R}_{\mathcal{F}}$ the class of fuzzy subsets of the real axis $u : \mathbb{R} \rightarrow [0, 1]$ satisfying the following properties:

- (i) u is normal, i.e., there exists $x_0 \in \mathbb{R}$ with $u(x_0) = 1$;
- (ii) u is a convex fuzzy set, i.e., $u(\lambda x + (1 - \lambda)y) \geq \min\{u(x), u(y)\}$ for all $\lambda \in [0, 1]$ and $x, y \in \mathbb{R}$;
- (iii) u is upper semicontinuous on \mathbb{R} , i.e. for each $a \in [0, 1]$, $\{x : u(x) < a\}$ is open in \mathbb{R} ;
- (iv) $cl\{x \in \mathbb{R} : u(x) > 0\}$ is compact, where cl denotes the closure of a set.

We call $\mathbb{R}_{\mathcal{F}}$ the space of fuzzy numbers.

Obviously $\mathbb{R} \subset \mathbb{R}_{\mathcal{F}}$. Here $\mathbb{R} \subset \mathbb{R}_{\mathcal{F}}$ is understood as $\mathbb{R} = \{\chi_{\{x\}}; x \text{ is usual real number}\}$. For $0 < r \leq 1$, denote $[u]^r = \{x \in \mathbb{R}; u(x) \geq r\}$ and $[u]^0 = cl\{x \in \mathbb{R}; u(x) > 0\}$.

By the definition of fuzzy numbers, it follows that for any $r \in [0, 1]$, $[u]^r$ is a bounded closed interval. The notation $[u]^r = [\underline{u}^r, \bar{u}^r]$ denotes explicitly the r -level set of u . We refer to \underline{u} and \bar{u} as the lower and upper branches on u , respectively.

For $u, v \in \mathbb{R}_{\mathcal{F}}$ and $\lambda \in \mathbb{R}$, the sum of two fuzzy numbers and the multiplication between a real and a fuzzy number are defined respectively by

$$[u + v]^r = [u]^r + [v]^r = \{x + y : x \in [u]^r, y \in [v]^r\}$$

and

$$[\beta u]^r = \beta[u]^r = \{\beta x : x \in [u]^r\}$$

for all $r \in [0, 1]$, where $[u]^r + [v]^r$ is the usual addition of two intervals of \mathbb{R} and $\beta[u]^r$ is the usual product of a number and a subset of \mathbb{R} .

The Hausdorff distance between fuzzy numbers is given by $d_\infty : \mathbb{R}_\mathcal{F} \times \mathbb{R}_\mathcal{F} \rightarrow \mathbb{R}^+ \cup \{0\}$ as $d_\infty(u, v) = \sup_{0 \leq r \leq 1} \max\{|\underline{u}^r - \underline{v}^r|, |\bar{u}^r - \bar{v}^r|\}$. The metric space $(\mathbb{R}_\mathcal{F}, d_\infty)$ is complete and the following properties for the metric d_∞ hold (see [15]):

- (i) $d_\infty(u + w, v + w) = d_\infty(u, v), \quad \forall u, v, w \in \mathbb{R}_\mathcal{F};$
- (ii) $d_\infty(u + w, v + z) \leq d_\infty(u, v) + d_\infty(w, z) \quad \forall u, v, w, z \in \mathbb{R}_\mathcal{F};$
- (iii) $d_\infty(ku, kv) = |k|d_\infty(u, v), \quad \forall k \in \mathbb{R}, u, v \in \mathbb{R}_\mathcal{F}.$

Definition 2.3. [17] A mapping $f : T \times \mathbb{R}_\mathcal{F} \rightarrow \mathbb{R}_\mathcal{F}$ is called continuous at $(t_0, x_0) \in T \times \mathbb{R}_\mathcal{F}$ provided for any arbitrary $\varepsilon > 0$, there exist an $\delta(\varepsilon) > 0$ such that

$$d_\infty(f(t, x), f(t_0, x_0)) < \varepsilon,$$

whenever $|t - t_0| < \delta(\varepsilon)$ and $d_\infty(x, x_0) < \delta(\varepsilon)$ for all $t \in T, x \in \mathbb{R}_\mathcal{F}$.

Definition 2.4. [6] Let $x, y \in \mathbb{R}_\mathcal{F}$. If there exists $z \in \mathbb{R}_\mathcal{F}$ such that $x = y + z$, then z is called the *H-difference* of x and y and it is denoted by $x \ominus y$.

Definition 2.5. [7, 19] Given $u, v \in \mathbb{R}_\mathcal{F}$, the *gH-difference* is the fuzzy number w , if it exists, such that

$$u \ominus_{gH} v = w \Leftrightarrow \quad (i) \ u = v + w \quad \text{or} \quad (ii) \ v = u + (-1) \cdot w \quad (1)$$

Remark 2.6. If $u \ominus_{gH} v$ exists, then its r -level set is given by

$$[u \ominus_{gH} v]^r = [\min \{\underline{u}^r - \underline{v}^r, \bar{u}^r - \bar{v}^r\}, \max \{\underline{u}^r - \underline{v}^r, \bar{u}^r - \bar{v}^r\}].$$

Definition 2.7. [1] The generalized Hukuhara derivative of a fuzzy-valued function $f : (a, b) \rightarrow \mathbb{R}_\mathcal{F}$ at x_0 is defined as

$$f'_{gH}(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h) \ominus_{gH} f(x_0)}{h}$$

If $f'_{gH}(x_0) \in \mathbb{R}_\mathcal{F}$, is said that f is *gH- differentiable* at x_0 .

Remark 2.8. If $f(x_0 + h) \ominus_{gH} f(x_0)$ exists as (i) in (1) is said that $f'_{gH}(x_0)$ is [(i) - gH]-differentiable and If it exists as (ii) in (1) is said that $f'_{gH}(x_0)$ is [(ii) - gH]- differentiable.

2.2 Fractional calculus

The Rimann-Liouville fractional derivative and integral of order $\alpha > 0$ of a function y , respectively are defined by (see [8])

$$D^\alpha y(t) := \frac{1}{\Gamma([\alpha] - \alpha)} \frac{d^{[\alpha]}}{dt^{[\alpha]}} \left(\int_0^t (t - s)^{[\alpha] - 1 - \alpha} y(s) ds \right) \quad (2)$$

and

$$I^\alpha y(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} y(s) ds, \quad (3)$$

besides, for $\alpha = 0$, we set $D^0 := I$, the identity operator. Also the Liouville-Caputo derivative is defined by

$${}^C D^\alpha y(t) := D^\alpha (y - T_{[\alpha]-1}[y])(t), \quad (4)$$

where $T_{[\alpha]-1}[y]$ is the Maclaurin polynomial of order $[\alpha] - 1$ of $y = y(t)$, i.e., $T_{[\alpha]-1}[y](t) = \sum_{k=0}^{m-1} \frac{y^{(k)}(0)}{k!} t^k$.

In the fractional calculus, the Mittag-Leffler function plays an important role same as the exponential function in the classical calculus. The Mittag-Leffler function of order $\alpha > 0$ is denoted by $E_\alpha(z)$ and it is defined as follows

$$E_\alpha(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha > 0, z \in \mathcal{C}. \quad (5)$$

Note that, for $\alpha = 1$ the function $E_1(z)$ becomes the classical exponential function $\exp(z)$. Moreover, by composing two functions $E_\alpha(z)$ and t^α we have

$$E_\alpha(\lambda t^\alpha) := \sum_{k=0}^{\infty} \frac{(\lambda t^\alpha)^k}{\Gamma(\alpha k + 1)}, \quad t \in [0, a] \subset \mathbb{R}, \quad (6)$$

where $\alpha > 0$ and $\lambda > 0$ is a constant. In fact, the function $E_\alpha(\lambda t^\alpha)$ is the unique solution to the following initial value problem

$$\begin{aligned} {}^C D^\alpha x(t) &:= D^\alpha (x - T_{[\alpha]-1}[x])(t) = \lambda x(t), \\ x(0) &= 1, x'(0) = 0, \dots, x^{[\alpha]-1}(0) = 0, \end{aligned} \quad (7)$$

for $t \geq 0$.

Here, it is necessary to review that for $\alpha \in \mathbb{N}$, $[\alpha] = \alpha$,

$${}^C D^\alpha f = D^\alpha (f - T_{\alpha-1}[f]) = D^\alpha f - D^\alpha (T_{\alpha-1}[f]) = D^\alpha f$$

since $T_{\alpha-1}[f]$ is a polynomial of degree $\alpha - 1$.

3 The main result

We start this section with the following definition of Liouville-Caputo's derivative under generalized Hukuhara derivative presented in [1].

Definition 3.1. Let $f_{gH}^{(m)} \in C[a, b] \cap L[a, b]$. The gH - fractional Liouville-Caputo differentiability of fuzzy valued function f is defined as follows:

$${}^C_{gH} D^\alpha f(x) = I^{m-\alpha} (f_{gH}^{([\alpha])})(x) = \frac{1}{\Gamma([\alpha] - \alpha)} \int_0^x (x-t)^{([\alpha]-\alpha-1)} f_{gH}^{([\alpha])}(t) dt \quad (8)$$

where $x > 0$ and $[\alpha] = m$.

Theorem 3.2. [1] Let $f : [a, b] \rightarrow \mathbb{R}_{\mathcal{F}}$ be a fuzzy-valued function on $[a, b]$.

- (i) If f is $[(i) - gH]$ -differentiable at $x_0 \in [a, b]$ then f is $^C[(i) - gH]$ - differentiable at x_0 .
- (ii) If f is $[(ii) - gH]$ -differentiable at $x_0 \in [a, b]$ then f is $^C[(ii) - gH]$ -differentiable at x_0 .

Lemma 3.3. [1] The Fuzzy Fractional Initial Value Problem (FFIVP)

$${}^C_{gH}D^\alpha(x(t)) = f(t, x(t)), \quad x(0) = x_0, \quad t \in [0, a], \tag{9}$$

for which $0 < \alpha < 1$, is equivalent to one of the following integral equations

$$x(t) = x(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (s - t)^\alpha f(s, x(s)) ds, \tag{10}$$

or

$$x(t) = x(0) \ominus (-1) \frac{1}{\Gamma(\alpha)} \int_0^t (s - t)^\alpha f(s, x(s)) ds. \tag{11}$$

Here, we introduce and define a new metric on space $\mathbb{R}_{\mathcal{F}}$

$$d_\infty^\lambda(u, v) = \max_{t \in [0, a]} \frac{d_\infty(u(t), v(t))}{E_\alpha(\lambda t^\alpha)}, \quad \alpha > 0, \lambda > 0, u, v \in \mathbb{R}_{\mathcal{F}}. \tag{12}$$

Lemma 3.4. Assume $d_\infty(u, v)$ is Hausdorff metric on $\mathbb{R}_{\mathcal{F}}$.

- (i) d_∞^λ is a metric on space of $C([0, a], \mathbb{R}_{\mathcal{F}})$.
- (ii) The metric d_∞^λ is equivalent to the metric $d_\infty^0 = \max_{t \in [0, a]} d_\infty(x(t), y(t))$.
- (iii) The metric space $((C[0, a], \mathbb{R}_{\mathcal{F}}), d_\infty^\lambda)$ is complete.

Proof. (i) If $\lambda > 0$ is constant, then we have $E_\alpha(\lambda t^\alpha) > 0$ for all $t \in [0, a]$ and E_α is continuous on $[0, a]$. Therefore, three properties of a metric can be easily verified.

- (ii) Since E_α is continuous and strictly increasing on $[0, a]$, we have

$$\frac{1}{E_\alpha(\lambda a^\alpha)} \leq \frac{1}{E_\alpha(\lambda t^\alpha)} \leq 1, \quad \text{for all } t \in [0, a], \tag{13}$$

hence, it ensures that two metrics d_∞^λ and d_∞^0 are equivalent.

- (iii) The completeness of $(C[0, a], d_\infty^\lambda)$ follows from the completeness of $(C[0, a], d_\infty^0)$ (see [10]) and (ii). If $\{x_n\}$ is a Cauchy sequence in $(C[0, a], d_\infty^0)$, thus (ii) follows that $\{x_n\}$ is a Cauchy sequence in $(C[0, a], d_\infty^\lambda)$ and

$$\lim_{n, m \rightarrow \infty} d_\infty^0(x_n, x_m) = 0, \quad \text{implies} \quad \lim_{n, m \rightarrow \infty} d_\infty^\lambda(x_n, x_m) = 0.$$

Q.E.D.

Lemma 3.5. [20] For every constant $\lambda > 0$, we have

$$\frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} E_\alpha(\lambda s^\alpha) ds = \frac{E_\alpha(\lambda t^\alpha) - 1}{\lambda} \quad \text{for all } t \geq 0. \tag{14}$$

Theorem 3.6. Let $R_0 = [0, a] \times \mathbb{R}_{\mathcal{F}}$ and $f : R_0 \rightarrow \mathbb{R}_{\mathcal{F}}$ be continuous such that the following Lipschitz condition holds: There exist a constant $L > 0$ such that

$$d_{\infty}(f(t, x), f(t, y)) \leq Ld_{\infty}(x, y), \quad (t, x), (t, y) \in R_0. \quad (15)$$

Then the FFIVP

$${}_gH^{\alpha}D^{\alpha}x(t) = f(t, x), \quad x(t_0) = x_0$$

has two unique solutions on $[0, a]$.

Proof. Since f is a continuous function on R_0 , so the integral equations (10) and (11) are well defined. Let $L > 0$ be the constant defined in (15), and set $\lambda := L\gamma$ where $\gamma > 1$ is an arbitrary constant. Consider the complete metric space $(C[0, a], d_{\infty}^{\lambda})$, and let operators F_1 and F_2 are defined by

$$(F_1x)(t) := x(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s)) ds, \quad t \in [0, a] \quad (16)$$

and

$$(F_2x)(t) := x(0) \ominus (-1) \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s)) ds \quad t \in [0, a]. \quad (17)$$

According to Lemma 3.3, the proof of the theorem is equivalent to show that each of two operators F_1 and F_2 have a fixed point, i.e. there exist unique points x_1 and x_2 such that $F_1(x_1) = x_1$ and $F_2(x_2) = x_2$.

So, we will prove that $F_1, F_2 : C([0, a], \mathbb{R}_{\mathcal{F}}) \rightarrow C([0, a], \mathbb{R}_{\mathcal{F}})$ are contractive maps with contraction constant $\sigma = 1/\gamma < 1$, with respect to the metric d_{∞}^{λ} .

For any $x, y \in C([0, a], \mathbb{R}_{\mathcal{F}})$, consider

$$\begin{aligned} d_{\infty}^{\lambda}(F_1x, F_1y) &:= \max_{t \in [0, a]} \frac{d_{\infty}(F_1x(t), F_1y(t))}{E_{\alpha}(\lambda t^{\alpha})} \\ &\leq \max_{t \in [0, a]} \left(\frac{1}{E_{\alpha}(\lambda t^{\alpha})} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} d_{\infty}(f(s, x(s)), f(s, y(s))) ds \right) \\ &\leq \max_{t \in [0, a]} \left(\frac{1}{E_{\alpha}(\lambda t^{\alpha})} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} Ld_{\infty}(x(s), y(s)) ds \right) \\ &= L \max_{t \in [0, a]} \left(\frac{1}{E_{\alpha}(\lambda t^{\alpha})} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} E_{\alpha}(\lambda s^{\alpha}) \frac{d_{\infty}(x(s), y(s))}{E_{\alpha}(\lambda s^{\alpha})} ds \right) \\ &\leq Ld_{\infty}^{\lambda}(x, y) \max_{t \in [0, a]} \left(\frac{1}{E_{\alpha}(\lambda t^{\alpha})} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} E_{\alpha}(\lambda s^{\alpha}) ds \right) \\ &= Ld_{\infty}^{\lambda}(x, y) \max_{t \in [0, a]} \left(\frac{1}{E_{\alpha}(\lambda t^{\alpha})} \left(\frac{E_{\alpha}(\lambda t^{\alpha}) - 1}{\lambda} \right) \right) \\ &= \frac{d_{\infty}^{\lambda}(x, y)}{\gamma} \max_{t \in [0, a]} \left(1 - \frac{1}{E_{\alpha}(\lambda t^{\alpha})} \right) \\ &= \frac{d_{\infty}^{\lambda}(x, y)}{\gamma} \left(1 - \frac{1}{E_{\alpha}(\lambda a^{\alpha})} \right) \\ &\leq \frac{d_{\infty}^{\lambda}(x, y)}{\gamma}. \end{aligned}$$

Thus, we see that the condition $\gamma > 1$ ensures that F_1x is a contractive map. Similarly we can prove that F_2x is a contractive map, so fuzzy fractional differential equation (9) has two unique solutions.

Q.E.D.

Remark 3.7. In Theorem 3.6, we use the metric (12) that involves the Mittag-Leffler function. This particular metric is optimal in the sense that it compels the operator to be contractive on the whole of $C([0, a], \mathbb{R}_{\mathcal{F}})$, i.e., the space of fuzzy continuous functions on $[0, a]$, rather than on a smaller set in the previous research.

For example, according to Theorem 4.3 in [1] it has been proved that a fuzzy fractional differential equation has two unique solutions on $[0, r]$, such that $0 \leq r \leq a$. In addition, the conditions of Theorem 4.3 [1] are somewhat robust whereas in this study, we only use the Lipschitz condition. In other words, the metric (12) not only help us to find two solutions on whole of $C([0, a], \mathbb{R}_{\mathcal{F}})$ but also it allows us to have an easier proof, that is we only need to provide the condition which satisfies the well-known Banach fixed-point theorem.

Example 3.8. suppose that $f(t, y(t)) = \eta y(t)$, so

$$d_{\infty}(f(t, y_1(t)), f(t, y_2(t))) \leq |\eta|d_{\infty}(y_1(t), y_2(t))$$

and $L = |\eta|$, since $\lambda := L\gamma$ and $\gamma > 1$, the fuzzy fractional differential equation ${}_gH^C D^{\alpha}x(t) = f(t, x(t))$, $x(0) = x_0$, where $x_0 \in \mathbb{R}_{\mathcal{F}}$, has two unique solutions on the space $C([0, a], \mathbb{R}_{\mathcal{F}})$ with the metric d_{∞}^{λ} for all $\lambda > |\eta|$. The exact solution of the fuzzy fractional differential equation for $\eta > 0$ in the case of ${}^C[(i) - gH]$ -differentiable is $x = x_0E_{\alpha}(\eta x^{\alpha})$ and for $\eta < 0$ in the case of ${}^C[(ii) - gH]$ -differentiable is $x = x_0E_{\alpha}(\eta x^{\alpha})$.

Example 3.9. Assume that $f(t, y(t)) := \frac{1}{\sqrt{\pi}}y(t) + (0, 1, 2)t$, thus

$$d_{\infty}(f(t, y_1(t)), f(t, y_2(t))) \leq \frac{1}{\sqrt{\pi}}d_{\infty}(y_1(t), y_2(t))$$

and $L = \frac{1}{\sqrt{\pi}}$, since $\lambda := L\gamma$ and $\gamma > 1$, the fuzzy fractional differential equation ${}_gH^C D^{\alpha}x(t) = f(t, x(t))$, $x(0) = [-0.1262r, -0.2523 + 0.1262r]$, $t \in [0, \frac{\pi}{2}]$, has two unique solutions on the space $C([0, \frac{\pi}{2}], \mathbb{R}_{\mathcal{F}})$ with the metric d_{∞}^{λ} for all $\lambda > \frac{1}{\sqrt{\pi}}$. For example, when $\alpha = 1$, the exact solution in the case of ${}^C[(i) - gH]$ -differentiable is,

$$\underline{x} = -\pi^2r - e^{\frac{t}{\pi}}\left(\frac{631r}{5000} - \pi^2r\right) - \pi rt,$$

$$\bar{x} = e^{\frac{t}{\pi}}\left(\frac{631r}{5000} - \pi^2r + 2\pi^2 - \frac{2523}{10000}\right) - 2\pi t + \pi^2r - 2\pi^2 + \pi rt$$

and in the case of ${}^C[(ii) - gH]$ -differentiable is

$$\underline{x} = e^{\frac{t}{\pi}}\left(\pi^2 - \frac{2523}{20000}\right) - \frac{\frac{631r}{5000} + \pi^2r - \pi^2 - \frac{2523}{20000}}{e^{\frac{t}{\pi}}} + \pi^2r - 2\pi^2 - \pi rt$$

$$\bar{x} = \frac{\left(\frac{631r}{5000} + \pi^2 r - \pi^2 - \frac{2523}{20000}\right)}{\frac{t}{e\pi}} - 2\pi t + e\frac{t}{\pi} \left(\pi^2 - \frac{2523}{20000}\right) - \pi^2 r + \pi r t.$$

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